MATH 121A Prep: Eigenvalues & Eigenvectors

Facts to Know: For a matrix A:

Eigenvalue: is a real number λ s.t. $A\vec{v} = \lambda\vec{v}$ has a non-zero solution

Eigenvector: corresponding to eigenvalue λ is a non-zero solution to $A\vec{v} = \lambda\vec{v}$.

Eigenspace: corresponding to eigenvalue λ is all eigenvalues + zero vector

Finding Eigenvalues: $A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} - \lambda Id\,\vec{v} = \vec{0} \Rightarrow (A - \lambda Id)\vec{v} = \vec{0}$
$\Leftrightarrow A - \lambda I$ is not invertible $\Leftrightarrow \boxed{\det(A - \lambda I) = 0}$ solve eqn for eigenvalues

Finding Eigenvectors: Solve $(A - \lambda I)\vec{v} = \vec{0}$ for a specific eigenvalue

Examples:

1. Find the eigenvalues of the matrix $\begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{bmatrix} = A$ Set $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 & -1 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & -2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 2\cdot 2 = -2 + 2\lambda - \lambda + \lambda^2 - 4$$
$$= \lambda^2 + \lambda - 6$$

$$0 = \det(A - \lambda I) = (1-\lambda)(\lambda^2 + \lambda - 6) = (1-\lambda)(\lambda+3)(\lambda-2)$$

$$\boxed{\lambda = 1,\ \lambda = -3,\ \lambda = 2}$$

2. Find the eigenvalues and corresponding eigenvectors for the matrix $\begin{bmatrix} 4 & 10 \\ 5 & -1 \end{bmatrix} = A$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 10 \\ 5 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 5 \cdot 10$$

$$-4 + \lambda - 4\lambda + \lambda^2 - 50 = \lambda^2 - 3\lambda - 54 = (\lambda - 9)(\lambda + 6) = 0$$

$\boxed{\lambda = 9}$ $\boxed{\lambda = -6}$ solutions to $(A - \lambda I)\vec{v} = \vec{0}$

$$\lambda = 9: \left[\begin{array}{cc|c} -5 & 10 & 0 \\ 5 & -10 & 0 \end{array}\right] \xrightarrow{R2 = R1 + R2} \left[\begin{array}{cc|c} -5 & 10 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R1 = -\frac{1}{5}R1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 Free $x_1 = 2x_2$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\lambda = -6: \left[\begin{array}{cc|c} 10 & 10 & 0 \\ 5 & 5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_2$ x_2 Free $x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

3. Explain why a matrix A is ^not invertible if and only if 0 is an eigenvalue of A.

λ is eigenvalue of A means $\det(A - \lambda I) = 0$

0 is eigenvalue of A $\Longleftrightarrow \det(A - 0\,Id) = 0$

$\Longleftrightarrow \det(A) = 0$

$\Longleftrightarrow A$ not invertible